Abel pairs and modular curves

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Abel pair

Definition

An *Abel pair* is a pair (X, α), where X is an complete algebraic curve and α a rational function on it, whose divisor has the form $\operatorname{div}(\alpha) = nA - nC$.

Example

Consider the family of elliptic curves $y^2 = (1 + kx)^2 - 4x^3$, with $k \in \mathbb{C} \setminus \{27^{1/3}\}$. Easy to check that $\alpha = 1 + kx - y$ is an Abel function on it.

How this is related to Abel? Abel described the type of elliptic integrals of the third kind that can be expressed in terms of elementary functions. These integrals can be described in terms of Abel pairs. For our example we obtain:

$$\int \frac{(3x+k)dx}{\sqrt{(x^2+kx)^2-4x}} = \ln(x(x+k)^2 - 2 + (x+k)\sqrt{(kx+x^2)^2-4x}) + C$$

Definition

We call Abel pairs $(\mathcal{X}_1, \alpha_1)$ and $(\mathcal{X}_2, \alpha_2)$ *isomorphic* if there exists an isomorphism $f : \mathcal{X}_1 \to \mathcal{X}_2$ and $\lambda \in PSL_2(\mathbb{C})$ such that $\lambda \circ \alpha_1 = \alpha_2 \circ f$.

$$\begin{array}{ccc} X_1 & \xrightarrow{f} & X_2 \\ {\scriptstyle \alpha_1}\big\downarrow & & {\scriptstyle \alpha_2}\big\downarrow \\ \mathbb{P}^1(\mathbb{C}) & \xrightarrow{\lambda} & \mathbb{P}^1(\mathbb{C}) \end{array}$$

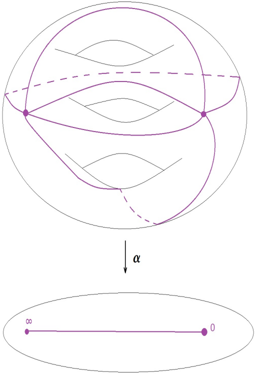
Actually, $\lambda : x \mapsto kx$ or $\lambda : x \mapsto k/x$. So $(\mathcal{X}, \alpha)$ and $(\mathcal{X}, 5\alpha)$, $(\mathcal{X}, 1/\alpha)$ are equal.

Definition

We call an Abel pair $(\mathcal{X}, \alpha)$ an *Abel-Belyi pair* if it is also a Belyi pair, i.e. α has only three critical values.

The dessin corresponding to an Abel-Belyi pair has one black vertex and one face.

Embedded graph, associated to a generic Abel pair



Generic means that $(\mathcal{X}, \alpha)$ has no critical values on $\mathbb{R}_{<0}$. Let $(\mathcal{X}, \alpha)$ be a generic Abel pair. $\Gamma_{X,\alpha} := \alpha^{-1}(\mathbb{R}_{\leq 0} \cup \{\infty\}) \subset X$ is a graph on the surface X.

Equation of Abel curves

Theorem

(i) $(\mathcal{X}, \alpha)$ *is an Abel pair of degree* n *if and only if there exist* $x \in \mathbb{C}[\mathcal{X}]$, $\deg x = k$, *and polynomials* P_1, $P_2, \ldots, P_{k-1}$, $\deg P_i \leq n$, *such that the equation for the curve* $\mathcal{X}$ *can be written as*

$$\alpha^k + \alpha^{k-1} P_{k-1}(x) + \cdots + \alpha P_1(x) + x^n = 0$$

(ii) If $\mathcal{X}$ *also is hyperelliptic than* $(\mathcal{X}, \alpha)$ *is an Abel pair of degree* n *if and only if there exist* $x \in \mathbb{C}[\mathcal{X}]$, $\deg x = 2$, *and polynomial* P, $\deg P \leq n$, *such that the equation for the curve* $\mathcal{X}$ *can be written as*

$$\alpha^2 + \alpha P(x) + x^n = 0$$

Theorem

(iii) If $(\mathcal{X}, \alpha)$ is an Abel pair of degree n on elliptic curve than there exist $x \in \mathbb{C}[\mathcal{X}]$, $\deg x = 2$, and polynomial P, $\deg P \leq \frac{n}{2}$, such that the equation for the curve $\mathcal{X}$ can be written as

$$\alpha^2 + \alpha P(x) + x^n = 0$$

Example

For $\mathcal{X} : y^2 = (1 + kx)^2 - 4x^3$ and $\alpha = \frac{1+kx-y}{2}$ we get equation of $\mathcal{X}$: $\alpha^2 + (1 + kx)\alpha + x^3 = 0$.

Abel pairs on elliptic curves and modular curves

Modular curve $X_1(n)$ parametrizes pairs $(E, A - B)$, where E is an elliptic curve and $A - B$ is a divisor of an order exactly n.
An Abel pair of genus 1 is actually determined by arbitrary elliptic curve and two points A, B, such that $n(A - B) \equiv 0$. But the true order of divisor $(A - B)$ may be less than n; it means that this Abel pair is a power of another Abel pair.

Definition

Abel pair $(\mathcal{X}, \alpha)$ is called *imprimitive* if there exists an other Abel pair $(\mathcal{X}, \alpha_0)$ and natural $k > 1$ such that $\alpha = \alpha_0^k$.

Theorem

$X_1(n)$ is isomorphic to the space of parameters of elliptic primitive Abel pair.

Belyi function on $X_1(n)$ defined by a family of Abel pairs

Let α is Abel function on elliptic curve. In general case α has two critical values besides 0 and ∞. Denote them k_1 and k_2. Then

Theorem

$\varkappa_n = \frac{\frac{k_1}{k_2}+\frac{k_2}{k_1}+2}{4}$ *is a well-defined clean Belyi function on* $X_1(n)$.

Next we describe dessin on $X_1(n)$ corresponding to $\varkappa_n$'s.

Example: $n = 3$.

- An Abel pair $(\mathcal{X}, \alpha)$ of degree 3 can be defined by the equation $F_k(x, \alpha) = \alpha^2 + (1 + kx)\alpha + x^3 = 0$.
- Easy to see that if $k_1 := k \cdot e^{2\pi/3}$, then $(\mathcal{X}_k, \alpha_k) \sim (X_{k_1}, \alpha_{k_1})$, so we consider $t := k^3$ as the parameter of this family of Abel pairs.
- Critical values of $\alpha_{\sqrt[3]{t}}$ are roots of the discriminant $F(x, \alpha)$ in x.

$$\Delta_x F(x, \alpha) = -\alpha^2(27\alpha^2 + (4t + 54)\alpha + 27)$$

  So critical values of $\alpha_{\sqrt[3]{t}}$ are 0, ∞ and roots of $27u^2 + (4t + 27)u + 27$.
- Hence we get $\varkappa_3 = \frac{\frac{u_1}{u_2} + \frac{u_2}{u_1} + 2}{4} = (\frac{2}{27}t + 1)^2$

Family of Abel pairs of degree $n = 5$

$n = 5$: $\alpha^2 + (t - (2t+1)x + (t+3)x^2)\alpha + x^5 = 0$

$\varkappa_5 = 1 - \frac{4}{5^8}\frac{(t^2+11t-1)(t+8)^2(3t+4)^3(3t-1)^3}{t^4}$

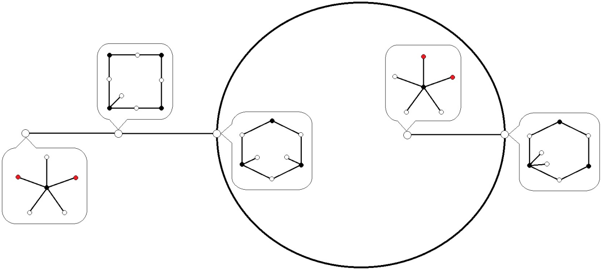


Рис.: dessin of $\varkappa_5$ on $X_1(5)$

Abel-Belyi pair on elliptic curve

Lemma

Dessins on the torus corresponding to the Abel-Belyi pairs have sets of valences $(n|n|3, 1, 1, \ldots, 1)$ *or* $(n|n|2, 2, 1, \ldots, 1)$.

These dessin can be described as

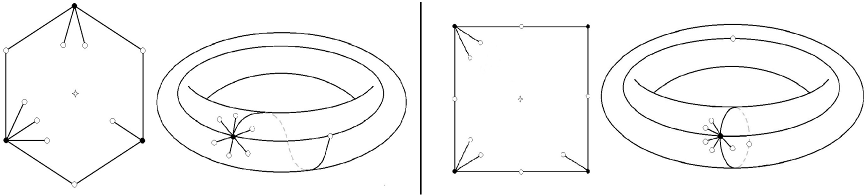


Рис.: $\langle 4, 2, 3\rangle$ and $\langle 3, 3, 2, 1\rangle$

Padé approximation

How these examples were calculated?

Definition

Padé approximation of order $[n, m]$ of a function $f(x)$ is the ratio of two polynomials $R_{[n,m]} = \frac{p_{[n,m]}(x)}{q_{[n,m]}(x)}$, $\deg p_{[n,m]}(x) \leq m$, $\deg q_{[n,m]}(x) \leq n$, for which $f^{(i)}(0) = R^{(i)}_{[n,m]}(0)$ for $0 \leq i \leq m + n$.

Lemma

Let $\frac{p_n(x)}{q_n(x)}$ *be a Padé approximation at 0 of the function* $\sqrt{1 + ax + bx^2 + cx^3}$, *where* $\deg p_n(x) = \left[\frac{n}{2}\right]$, $\deg q_n(x) = \left[\frac{n-3}{2}\right]$. *The function* $\varphi_n = p_n(x) - q_n(x)y$ *on* X *defined by* $y^2 = 1 + ax + bx^2 + cx^3$ *has a divisor* $\mathrm{div}(\varphi_n) = (n-1)A + B - nC$.

The coefficients of p, q are linear functions in the coefficients of Taylor series for $\sqrt{1 + ax + bx^2 + cx^3}$. The condition $A = B$ gives the needed equation defining $X_1(n)$.

Some calculated examples

$n=4$	$\varkappa_4$	$\frac{1}{64}\ \frac{(t-1)(t-9)^3}{t^3}$
$n=5$	$\varkappa_5$	$\frac{4}{5^{10}}\ \frac{\left(-1+11\,a+a^2\right)(a+8)^2(3\,a+4)^3(3\,a-1)^3}{a^4}$
$n=6$	$\varkappa_6$	$\frac{1}{2^{14}\cdot 3^{12}}\ \frac{(9\,a-1)(9\,a-25)^2\left(81\,a^3-27\,a^2+99\,a-25\right)^3}{a^5(a-1)^4}$
$n=7$	$\varkappa_7$	$\frac{2^2}{7^{14}}\ \frac{(l^3-8\,l^2+5\,l+1)(l-5)^2(4\,l^2-26\,l-5)^2(2\,l^2+l+1)^2(3\,l^2-9\,l+5)^3(12\,l^3-12\,l^2+4\,l+5)^3}{l^{10}(l-1)^6}$
$n=8$	$\varkappa_8$	$\frac{1}{2^{50}}\ \frac{P_8}{(-1+l)^6(1+l)^4 l^7}$
$n=9$	$\varkappa_9$	$\frac{1}{2^2\cdot 3^{36}}\ \frac{P_9}{v^8(v^2-v+1)^6(v-1)^{14}}$
$n=10$	$\varkappa_{10}$	$\frac{2^4}{5^{20}}\ \frac{P_{10}}{u^8(u^2+4\,u-1)^5(u+1)^9(u-1)^{21}}$
$n=12$	$\varkappa_{12}$	$\frac{1}{2^{50}\cdot 3^{24}}\ \frac{P_{12}}{v^{11}(v+1)^6(v^2+1)^8(v^2-v+1)^9(v-1)^{10}}$

$P_8=(l^2-6\,l+1)(2401+294\,l-1881\,l^2+564\,l^3+575\,l^4-250\,l^5+25\,l^6)^2(225\,l^6-450\,l^5-225\,l^4+36\,l^3-81\,l^2-130\,l+49)^3$

$P_9=(1-6\,v+3\,v^2+v^3)(5\,v^3+15\,v^2-3\,v+32)^2(40\,v^9-576\,v^7+1272\,v^6+432\,v^5-6525\,v^4+13521\,v^3-14229\,v^2+8064\,v-2048)^2(5\,v^3+3\,v-1)^3(20\,v^6-60\,v^5+48\,v^4+40\,v^3-96\,v^2+57\,v-16)^3$

$P_{10}=(u^2-u-1)(54\,u^{14}+162\,u^{13}-1854\,u^{12}+333\,u^{11}+31324\,u^{10}-88744\,u^9-67532\,u^8+944894\,u^7-2380538\,u^6+2801086\,u^5-1678854\,u^4+595213\,u^3-133896\,u^2+17792\,u-1024)^2(9\,u^{10}+45\,u^9+40\,u^8-150\,u^7-430\,u^6-884\,u^5-360\,u^4-70\,u^3+25\,u^2+15\,u-4)^3$

$P_{12}=(v^2-4\,v+1)(214358881-496037080\,v+60025\,v^{26}-960400\,v^{25}-202855541\,v^{18}+131268550\,v^{20}-41239576\,v^{19}+553750568\,v^{17}+64706950\,v^{22}-114532600\,v^{21}+6542725\,v^{24}-25450600\,v^{23}+1746882820\,v^{12}-75890780\,v^{14}-903401008\,v^{13}-1872752176\,v^{11}+1106930815\,v^{10}+404345024\,v^9-808291433\,v^{16}+697152752\,v^{15}-1613955101\,v^8+2276564984\,v^7-1614356618\,v^6+612680360\,v^5+604125910\,v^4-1011984952\,v^3+1002337501\,v^2)^2(1225\,v^{14}-4900\,v^{13}+8575\,v^{12}-9800\,v^{11}+6125\,v^{10}-1100\,v^9-2325\,v^8+336\,v^7-1317\,v^6+4\,v^5+605\,v^4-1160\,v^3+943\,v^2-532\,v+121)^3$

Primitive elliptic Abel-Belyi pairs

What elliptic Abel-Belyi pairs are primitive?

Theorem

(i) A toric dessin $\langle a, b, c \rangle$ corresponds to a Belyi function, which is an m-th power if and only if m divides $gcd(a, b, c)$.

(ii) A toric dessin $\langle a, b, c, d \rangle$ corresponds to a Belyi function, which is an m-th power if and only if $a \equiv -b \equiv c \equiv -d \pmod{m}$.

Example

Toric dessin $\langle n, n, n \rangle$ corresponds to Belyi pair with $\mathcal{X} : y^2 = 1 - x^3$, $\beta = \left(\frac{1-y}{2}\right)^n$.

Number of Abel-Belyi pairs on $X_1(n)$

Theorem

If $n > 3$ then

(i) The number of primitive Abel-Belyi pairs of type $\langle a, b, c\rangle$ is

$$m_3(n) = \frac{\varphi(n)\psi(n)}{6} - \frac{\varphi(n)}{2}$$

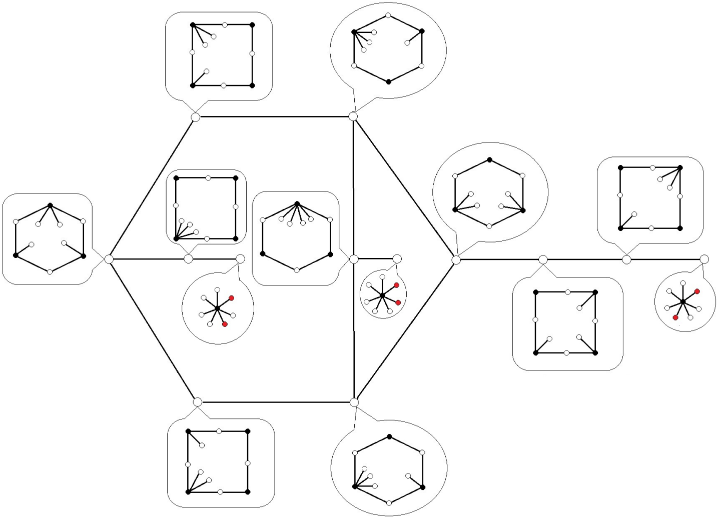
(ii) The number of primitive Abel-Belyi pairs of type $\langle a, b, c, d\rangle$

$$m_2(n) = \frac{(n-6)\varphi(n)\psi(n)}{24} + \frac{\varphi(n)}{2}$$

where $\varphi(n)$ is the Euler function and $\psi(n)$ is the Dedekind psi function $\psi(n) = n\prod_{p|n}\left(1 + \frac{1}{p}\right)$.

$n = 7$

dessin on $X_1(7)$ corresponds to $\varkappa_7$

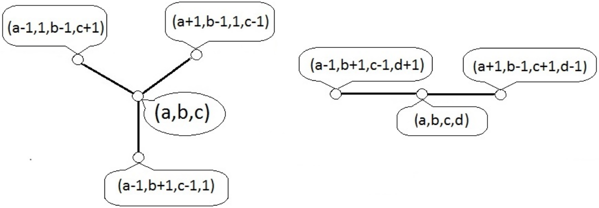


Generic pictures of the dessin on $X_1(n)$

Theorem

(i) For $n > 3$ the dessin on $X_1(n)$ corresponding to $\varkappa_n\ \langle a, b, c\rangle$ has valence 3 and $\langle a, b, c, d\rangle$ has valence 2.

(ii) They are connected with the dessins



If one of the numbers in these formulas vanishes, it should be thrown out. The vanishing of two numbers implies decreasing of the genus.

Thank you!